

# SPECTRAL, STOCHASTIC AND CURVATURE ESTIMATES FOR SUBMANIFOLDS OF HIGHLY NEGATIVE CURVED SPACES

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**ABSTRACT.** We prove spectral, stochastic and mean curvature estimates for complete  $m$ -submanifolds  $\varphi: M \rightarrow N$  of  $n$ -manifolds with a pole  $N$  in terms of the comparison isoperimetric ratio  $I_m$  and the extrinsic radius  $r_\varphi \leq \infty$ . Our proof holds for the bounded case  $r_\varphi < \infty$ , recovering the known results, as well as for the unbounded case  $r_\varphi = \infty$ . In both cases, the fundamental ingredient in these estimates is the integrability over  $(0, r_\varphi)$  of the inverse  $I_m^{-1}$  of the comparison isoperimetric radius. When  $r_\varphi = \infty$ , this condition is guaranteed if  $N$  is highly negatively curved.

## 1. INTRODUCTION

Let  $N$  be a complete Riemannian  $n$ -manifold with a pole  $p$  and let  $\varphi: M \rightarrow N$  be an isometric immersion of a complete Riemannian  $m$ -manifold  $M$  into  $N$ . The extrinsic radius  $r_\varphi$  is the (possibly extended) number

$$r_\varphi = \inf\{r \in (0, \infty] : \varphi(M) \subset B_N(r)\},$$

where  $B_N(r) \subseteq N$  is the geodesic ball centered at  $p$  of  $N$  with radius  $r \in (0, \infty]$ . Suppose that the radial<sup>1</sup> sectional curvatures of  $N$  at  $x \in B_N(r_\varphi)$  satisfies

$$(1) \quad K_N(x) \leq -G(\rho_N(x))$$

where  $G: \mathbb{R} \rightarrow \mathbb{R}$  is a smooth even function and  $\rho_N(x) = \text{dist}_N(p, x)$ .

Associate to  $N$ , the  $m$ -dimensional *model* manifold  $\mathbb{M}_\sigma^m$  with radial sectional curvature  $-G(r)$ . Namely,

$$\mathbb{M}_\sigma^m = ([0, r_\varphi] \times \mathbb{S}^{m-1}, ds_\sigma^2 = dr^2 + \sigma^2(r) d\theta^2),$$

where  $\sigma$  denotes the unique solution of the Cauchy problem on  $(0, r_\varphi]$

$$(2) \quad \begin{cases} \sigma''(t) - G(t)\sigma(t) = 0, \\ \sigma(0) = 0, \quad \sigma'(0) = 1, \end{cases}$$

which we assume to be positive and increasing on  $[0, r_\varphi)$ . Observe that if  $G \geq 0$  on  $[0, \infty)$  then  $\sigma'' \geq 0$  everywhere and  $\sigma' \geq 1 > 0$  on  $[0, \infty)$ . More generally, it follows

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<sup>1</sup>Along the geodesics issuing from  $p$ .

by a strengthened version of Kneser Theorem, see e.g. [10, Prop. 1.21], that if  $G_- = \max\{-G, 0\}$  satisfies

$$(3) \quad t \int_t^\infty G_-(s) ds \leq \frac{1}{4},$$

then  $\sigma' \geq 0$ .

Associate to the model  $\mathbb{M}_\sigma^m$  the function  $I_m: [0, r_\varphi] \rightarrow \mathbb{R}_+$  defined by

$$I_m(r) = \frac{\sigma(r)^{m-1}}{\int_0^r \sigma(t)^{m-1} dt}.$$

In geometric terms,  $I_m$  is the *non-homogeneous isoperimetric ratio*

$$I_m(r) = \frac{\text{vol } \partial B_r(o)}{\text{vol } B_r(o)},$$

where  $B_r(o)$  and  $\partial B_r(o)$  are the geodesic balls and spheres in  $\mathbb{M}_\sigma^m$  of radius  $r > 0$  and center at the pole  $o$  of the model. In particular, the Cheeger constant  $h(\mathbb{M}_\sigma^m)$  of the model has the upper estimate

$$\inf_{[0, +\infty)} I_m(r) \geq h(\mathbb{M}_\sigma^m).$$

Also associated to the model  $\mathbb{M}_\sigma^m$ , is the *homogeneous isoperimetric ratio*  $\mathcal{I}_m(r)$  which is defined by

$$\mathcal{I}_m(r) = \frac{\sigma(r)^m}{\int_0^r \sigma(t)^{m-1} dt} = \frac{\text{vol } \partial B_r(o)^{\frac{m}{m-1}}}{\text{vol } B_r(o)}.$$

We showed in [8] that if  $\varphi: M \rightarrow N$  is minimal and the following extrinsic conditions

$$\mathcal{I}_m'(t) \geq 0, \quad t \in (0, r_\varphi) \quad \text{and} \quad I_m^{-1} \in L^1(0, r_\varphi)$$

hold, then the global mean exit time of  $M$  is finite, and, in fact,

$$E_M \leq \int_0^{r_\varphi} I_m^{-1}(s) ds < \infty.$$

In particular,  $M$  is not  $L^1$ -Liouville. Three aspects of this result should be remarked. First, the condition  $\mathcal{I}_m'(t) \geq 0$  is implied by  $-G \leq 0$ . Indeed,

$$\mathcal{I}_m'(t) = m \frac{\sigma'}{\sigma} - \frac{\sigma^{m-1}}{\int_0^t \sigma^{m-1}} = \frac{1}{\sigma \int_0^t \sigma^{m-1}} \left( m \sigma' \int_0^t \sigma^{m-1} - \sigma^m \right),$$

thus  $\mathcal{I}_m'(t) \geq 0 \Leftrightarrow (m \sigma' \int_0^t \sigma^{m-1} - \sigma^m) \geq 0$ . Now,  $(m \sigma' \int_0^t \sigma^{m-1} - \sigma^m) \rightarrow 0$  as  $r \rightarrow 0+$  and its derivative is

$$m \sigma'' \int_0^t \sigma^{m-1}.$$

Thus if  $\sigma'' \geq 0$ , that is, if the curvature is nonpositive, then  $m \sigma'' \int_0^t \sigma^{m-1} \geq 0$  and therefore  $\mathcal{I}_m'(t) \geq 0$ . Note however that, while it is necessary that  $\sigma'' \geq 0$  in a right neighborhood of 0,  $\mathcal{I}_m(t)$  could be nondecreasing even in the presence of some controlled negativity of  $\sigma''$ .

Second, the requirement  $I_m^{-1} \in L^1(0, r_\varphi)$  is automatically satisfied if  $r_\varphi < \infty$ , and in this case, in [8, Thm.9] we recover S. Markvorsen's result [15, Thm.1-item ii.] under the slightly weaker hypothesis  $\mathcal{S}'_m(t) \geq 0$ .

Finally, in the case where the extrinsic diameter is infinite, the condition that  $I_m^{-1}$  is integrable is equivalent to the stochastic incompleteness and implies a great amount of negative curvature of the  $m$ -dimensional model  $\mathbb{M}_\sigma^m$  (and thus of  $N$ ).

The result [8, Thm.9] suggests that there should exist a correspondence between results valid for complete, bounded submanifolds of  $N$  and companion results for complete, unbounded immersions  $\varphi: M \rightarrow N$ , into a manifold  $N$  with a pole, with radial sectional curvatures bounded above as in (1) and such that  $I_m^{-1} \in L^1(0, +\infty)$ .

The purpose of this paper is to show that this correspondence does exist for a variety of results, including well known curvature, stochastic and spectral estimates for bounded submanifolds, of which we shall prove counterparts in the unbounded highly negatively curved setting.

## 2. STATEMENT OF THE RESULTS

**Theorem 1** (Spectral estimates). *Let  $\varphi: M \rightarrow N$  be an isometric immersion of a complete  $m$ -dimensional Riemannian manifold  $M$  into the complete  $n$ -dimensional Riemannian manifold  $N$  with a pole  $p \in N$ . Let  $\rho_N(y) = \text{dist}_N(p, y)$  and assume that the radial sectional curvature of  $N$  satisfies*

$$\text{Sec}_{\text{rad}}^N(x) \leq -G(\rho_N(x))$$

*for some smooth, even function  $G$ . Assume also that the solution  $\sigma$  of (2) satisfies  $\sigma' \geq 0$  on  $[0, \text{diam}\varphi(M)]$ , that  $\mathcal{S}_m$  is nondecreasing in that interval, and*

$$A = \sup_M \frac{|\mathbf{H}|(x)}{I_m(\rho_N(\varphi(x)))} \leq 1,$$

*where  $\mathbf{H}$  is the mean curvature vector field of  $\varphi$ . Then*

- (a) *The bottom of the spectrum of the Laplace-Beltrami operator of  $M$  satisfies the estimate*

$$\lambda^*(M) \geq \max \left\{ \frac{(1-A)}{\int_0^{r_\varphi} I_m(t)^{-1} dt}, (1-A)^2 \inf_{[0, r_\varphi]} \frac{I_m(r)^2}{4} \right\}$$

- (b) *If  $\varphi$  is proper in the geodesic ball  $B_N(r_\varphi)$ ,  $\int_0^{r_\varphi} I_m^{-1} dt < +\infty$  and  $A < 1$  then the spectrum of  $-\Delta_M$  is discrete.*

Observe that item (b) of Theorem 1 extends the main result of [6]. It is worth noticing that in the case where  $\sigma'/\sigma$  is nonincreasing, then  $I'_m(r) < 0$ . Indeed, it is easy to check that  $I_m$  satisfies the Riccati equation

$$I'_m + I_m^2 = (m-1) \frac{\sigma'}{\sigma} I_m,$$

and that  $I_m(r) \sim m/r$  as  $r \rightarrow 0$ . It follows that  $I'_m < 0$  and  $I_m > (m-1)\sigma'/\sigma$  in a right neighborhood of 0, and an easy comparison argument shows that if the right

hand side of the Riccati equation is nonincreasing then  $I'_m < 0$  where defined. In particular,

$$\inf_{[0, r_\varphi]} I_m = \lim_{r \rightarrow r_\varphi} I_m(r) \geq \lim_{r \rightarrow r_\varphi} (m-1) \frac{\sigma'}{\sigma}.$$

In the special case where  $K_N(x) \leq -k^2 < 0$ , so that  $\mathbb{M}_\sigma^m = \mathbb{H}_{-k^2}^m$  is hyperbolic space of curvature  $-k^2$ , we have

$$\inf_{[0, r_\varphi)} I_m = \begin{cases} (m-1)k \coth(kr_\varphi) & \text{if } r_\varphi < +\infty, \\ (m-1)k & \text{if } r_\varphi = +\infty \end{cases}$$

Thus, if  $r_\varphi = +\infty$  and  $|\mathbf{H}| \leq H_o < (m-1)k$ , we have

$$(1-A)^2 \inf_{[0, +\infty)} \frac{I_m(r)^2}{4} = \left(1 - \frac{H_o}{(m-1)k}\right)^2 \frac{(m-1)^2 k^2}{4} = \frac{[(m-1)k - H_o]^2}{4},$$

and, in particular, we recover a result by L.-F. Cheung and P.-F. Leung, [11, Theorem 2], and Bessa and Montenegro, [4, Corollary 4.4]. Similarly, in the case where  $r_\varphi < +\infty$ , if  $k > 0$   $H_o < (m-1)k \coth(kr_\varphi)$ , then

$$(1-A)^2 \inf_{[0, +\infty)} \frac{I_m(r)^2}{4} = \frac{[(m-1)k \coth(kr_\varphi) - H_o]^2}{4},$$

while if  $k = 0$  and  $H_o < (m-1)/r_\varphi$ , then

$$(1-A)^2 \inf_{[0, +\infty)} \frac{I_m(r)^2}{4} = \frac{[(m-1)/r_\varphi - H_o]^2}{4}.$$

and we recover results by K. Seo [17].

A closer inspection of the proof of Theorem 1 shows that if we let  $M = N$ , then the conclusion holds without having to assume that  $\mathcal{J}_n$  be increasing. Thus we have

**Corollary 2.** *Let  $N$  be a complete Riemannian  $n$ -manifold with a pole  $p$  and radial sectional curvature satisfying*

$$\text{Sec}_{\text{rad}}^N(x) \leq -G(\rho_N(x))$$

where  $G: \mathbb{R} \rightarrow \mathbb{R}$  is a smooth even function and the solution  $\sigma$  of the initial value problem (2) satisfies  $\sigma' \geq 0$  on  $[0, +\infty)$ . Then

$$(4) \quad \lambda^*(N) \geq \lambda^*(\mathbb{M}_\sigma^n) \geq \max \left\{ \frac{[\inf_{(0, \infty)} I_n(r)]^2}{4}, \frac{1}{\int_0^\infty I_n^{-1}(r) dr} \right\}.$$

Moreover, if  $\int_0^{+\infty} I_n^{-1} dt < +\infty$ , then the spectrum of  $N$  is purely discrete.

Observe that both alternatives do occur:

$$\frac{1}{\int_0^{r_\varphi} I_m(t)^{-1} dt} > \inf_{[0, r_\varphi)} \frac{I_m(r)^2}{4}$$

or

$$\frac{1}{\int_0^{r_\varphi} I_m(t)^{-1} dt} < \inf_{[0, r_\varphi)} \frac{I_m(r)^2}{4}$$

as shown by the examples below. Indeed, consider the 2-dimensional model  $\mathbb{M}_\sigma^2$ , with

$$\sigma(t) = \left(r + \frac{r^7}{2}\right) \exp \frac{r^6}{6}.$$

It is easy to show that

$$\frac{1}{\int_0^\infty I_2(s)^{-1} ds} = \frac{3 \cdot 2^{2/3} \sqrt{3}}{\pi} \approx 2.62 > \inf_{[0, \infty)} \frac{I_2(r)^2}{4} = \frac{36}{25} \cdot \left(\frac{2}{3}\right)^{-1/3} \approx 1.64.$$

On the other hand, if  $\mathbb{M}_\sigma^m = \mathbb{H}^m(-1)$  is a totally geodesic hyperbolic space in  $N = \mathbb{H}^n(-1)$ , then

$$\frac{1}{\int_0^\infty I_m(s)^{-1} ds} = 0$$

by stochastic completeness, while, as observed above,

$$\inf_{[0, \infty)} \frac{I_m(r)^2}{4} = \frac{(m-1)}{4}.$$

We next describe mean curvature estimates which extend previous results valid for bounded immersions obtained, in increasing generality, in [2], [12], [13], [14], and [16].

**Theorem 3** (Mean curvature estimates). *Let  $\varphi : M \rightarrow N$  be an isometric immersion of a stochastically complete,  $m$ -dimensional Riemannian manifold  $M$  into a  $n$ -dimensional Riemannian manifold  $N$  with a pole  $o \in N$ . Assume that the radial sectional curvature of  $N$  satisfies*

$$\text{Sec}_{\text{rad}}^N(x) \leq -G(\rho_N(x))$$

for some smooth, even function  $G(t)$  satisfying (3). Assume also that

- (i)  $\mathcal{I}_m(r)$  is non-decreasing for  $r \in (0, r_\varphi]$
- (ii)

$$I_m(r)^{-1} \in L^1(0, r_\varphi).$$

Then

$$\sup_M \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \geq 1$$

In particular, if  $r_\varphi = +\infty$ , ( $\varphi$  is unbounded in  $N$ ) then

$$\limsup_{x \rightarrow \infty} \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \geq 1.$$

If  $r_\varphi = +\infty$  and  $I_m(r)^{-1} \rightarrow 0$  as  $r \rightarrow +\infty$  then

$$\sup_M |H| = +\infty.$$

**Remark 4.** *If the mean curvature  $H$  of  $M$  is bounded, then either  $r_\varphi < +\infty$ , and  $\varphi$  has bounded image in  $N$ , or  $I_m(r)^{-1} \not\rightarrow 0$  as  $r \rightarrow +\infty$ . An immediate consequence is that, under the above assumptions, an immersed submanifold with bounded mean curvature in  $N$  is stochastically incomplete. This completes the picture initiated in*

[8, Thm.10], where we proved that, regardless the condition (ii) on the isoperimetric ratio  $I_m(r)$ , a properly immersed minimal submanifold of  $N$  is not  $L^1$ -Liouville (hence stochastically incomplete).

We also note that, according to Theorem 3, Theorem 1 does not apply if  $I_m^{-1}$  is integrable and  $M$  is stochastically complete.

**Remark 5.** An application of de L'Hospital rule show that condition  $I_m(r)^{-1} \rightarrow 0$  as  $r \rightarrow \infty$  holds provided  $\frac{\sigma'}{\sigma} \rightarrow +\infty$ , which in turn is typical of a super-exponential behavior of  $\sigma(r)$ . We have already remarked that condition  $\mathcal{I}_m' \geq 0$  is satisfied provided  $\sigma''(r) \geq 0$ . Finally, as already recalled, the integrability of  $I_m(r)^{-1}$  is equivalent to the stochastic incompleteness of the model  $\mathbb{M}_\sigma^m$  and is implied by a sufficiently fast growth of  $\sigma$ .

### 3. PRELIMINARIES

Let  $M$  be a smooth Riemannian manifold and  $\Omega \subset M$  an arbitrary open subset. The fundamental tone  $\lambda^*(\Omega)$  of  $\Omega$ , is defined by

$$\lambda^*(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}, f \in H_0^1(\Omega) \setminus \{0\} \right\},$$

where  $H_0^1(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  with respect to the norm

$$\|\varphi\|_\Omega^2 = \int_\Omega \varphi^2 + \int_\Omega |\nabla \varphi|^2.$$

When  $\Omega = M$  is a complete non-compact Riemannian manifold, the fundamental tone  $\lambda^*(M)$  coincides with the bottom  $\inf \Sigma(-\Delta)$  of the  $L^2$ -spectrum  $\Sigma(-\Delta) \subset [0, \infty)$  of the unique self-adjoint extension of the Laplacian  $\Delta$  acting on  $C_0^\infty(M)$  also denoted by  $\Delta$ . When  $\Omega$  is compact with boundary  $\partial\Omega$ , then the fundamental tone is the bottom of the  $L^2$ -spectrum of the Friedrichs extension of  $-\Delta$  initially defined on  $C_c^\infty(\Omega)$ . Moreover, there exists  $u \in C^\infty(\Omega) \cap H_0^1(\Omega)$ , positive in  $\Omega$  satisfying  $\Delta u + \lambda^*(\Omega)u = 0$ , ( $u|_{\partial\Omega} = 0$  if  $\partial\Omega \neq \emptyset$  and piecewise smooth). The spectrum decomposes as  $\Sigma(-\Delta) = \Sigma_p(-\Delta) \cup \Sigma_{ess}(-\Delta)$  where  $\Sigma_p(-\Delta)$  is formed by eigenvalues with finite multiplicity and  $\Sigma_{ess}(-\Delta)$  is formed by accumulation points of the spectrum and by the eigenvalues with infinite multiplicity. It is said that  $M$  has discrete spectrum if  $\Sigma_{ess}(-\Delta) = \emptyset$  and that  $M$  has purely continuous spectrum if  $\Sigma_p(-\Delta) = \emptyset$ . It is well known that for every exhaustion of  $M$  by relatively compact open sets  $\{K_j\}$  with boundary, one has,  $\inf \Sigma_{ess}(-\Delta) = \lim_{j \rightarrow +\infty} \lambda^*(M \setminus K_j)$ . It follows that  $-\Delta$  has pure discrete spectrum if and only if

$$\lim_{j \rightarrow +\infty} \lambda^*(M \setminus K_j) = \infty.$$

The following two lemmas are useful to obtain lower bounds for the fundamental tones of open sets of Riemannian manifolds.

**Lemma 6** ([4]). *Let  $\Omega \subset M$  be an open subset of a Riemannian manifold  $M$ . Then the fundamental tone of  $\Omega$  is bounded below by*

$$(5) \quad \lambda^*(\Omega) \geq \frac{c(\Omega)^2}{4},$$

where  $c(\Omega) = \sup \left\{ \frac{\inf_{\Omega} \operatorname{div} X}{\sup_{\Omega} |X|} : X \in \mathcal{X}^\infty(\Omega), \operatorname{div} X \geq 0 \right\}$  and  $\mathcal{X}^\infty(\Omega)$  is the set of all smooth vector fields in  $\Omega$ .

**Lemma 7** (Barta [3], [5]). *Let  $\Omega \subset M$  be an open subset of a Riemannian manifold  $M$  and  $u : \Omega \rightarrow \mathbb{R}$  be a smooth positive function. Then*

$$\lambda^*(\Omega) \geq \inf_{\Omega} \left[ -\frac{\Delta u}{u} \right].$$

#### 4. PROOF OF THE RESULTS

*Proof of Theorem 1.* Recall that if  $\varphi : M \rightarrow N$  is an isometric immersion and  $g : N \rightarrow \mathbb{R}$  and  $F : \mathbb{R} \rightarrow \mathbb{R}$  are smooth functions, then for every  $X \in T_x M$  we have

$$\begin{aligned} \operatorname{Hess}(F \circ g \circ \varphi)(X, X) &= F''(g(\varphi(x))) \langle \nabla^N g, d\varphi X \rangle^2 \\ &\quad + F'(g(\varphi(x))) [\operatorname{Hess}^N g(d\varphi X, d\varphi X) + \langle \nabla^N g, II(X, X) \rangle]. \end{aligned}$$

If  $g = \rho_N$  is the distance function, then the assumption on the sectional curvature of  $N$  implies

$$\operatorname{Hess}_N \rho_N(Y, Y) \geq \frac{\sigma'}{\sigma} [\langle Y, Y \rangle - \langle \nabla^N \rho_N, Y \rangle^2]$$

on  $B_N(r_\varphi)$ .

Assuming that  $F' \geq 0$ , letting  $\{X_i\}_{i=1}^m$  be an orthonormal basis of  $T_x M$  and setting  $\rho_x = \rho_N(\varphi(x))$ , we obtain

$$\begin{aligned} \Delta(F \circ \rho \circ \varphi)(x) &\geq m(F' \frac{\sigma'}{\sigma})(\rho_x) + (F'' - F' \frac{\sigma'}{\sigma})(\rho_x) \sum_{i=1}^m \langle \nabla^N \rho_N, d\varphi X_i \rangle^2 \\ &\quad + F'(\rho_x) \langle \nabla^N \rho_N, \mathbf{H} \rangle. \end{aligned}$$

Let

$$(6) \quad F(r) = \int_0^r \frac{\int_0^t \sigma^{m-1}(s) ds}{\sigma^{m-1}(t)} dt,$$

so that

$$F'(r) = \frac{\int_0^r \sigma^{m-1}(s) ds}{\sigma^{m-1}(r)} = I_m^{-1}(r) > 0 \text{ and } F''(r) = 1 - (m-1) \frac{\sigma'}{\sigma} F'(r),$$

which, inserted into the last inequality, yield

$$\begin{aligned} \Delta(F \circ \rho \circ \varphi)(x) &\geq m(F' \frac{\sigma'}{\sigma})(\rho_x) + \left[ (1 - m \frac{\sigma'}{\sigma} F'_R(\rho_x)) \sum_{i=1}^m \langle \nabla^N \rho, d\varphi X_i \rangle^2 \right. \\ &\quad \left. - F'(\rho_x) |H(\varphi(x))| \right] \end{aligned}$$

We complete  $\{d\varphi X_i\}_{i=1}^m$  to an orthonormal basis  $\{d\varphi X_i\}_{i=1}^m \cup \{Y_j\}_{j=m+1}^n$  on  $T_{\varphi(x)}N$ , and note that

$$\sum_i \langle \nabla^N \rho, d\varphi X_i \rangle^2 + \sum_j \langle \nabla^N \rho, Y_j \rangle^2 = 1.$$

Inserting this into the above inequality and using the assumption  $\mathcal{J}'_m(t) \geq 0$  in the form

$$m \frac{\sigma'}{\sigma} F' = m \frac{\sigma'}{\sigma} \frac{\int_0^t \sigma^{m-1}(s) ds}{\sigma^{m-1}(t)} \geq 1$$

we finally obtain

$$\begin{aligned} \Delta(F \circ \rho \circ \varphi)(x) &\geq 1 + \left[ m \left( F' \frac{\sigma'}{\sigma} \right)(\rho_x) - 1 \right] \sum_j \langle \nabla^N \rho, Y_j \rangle^2 - F'(\rho_x) |\mathbf{H}(\varphi(x))| \\ (7) \quad &\geq 1 - F'(\rho_x) |\mathbf{H}(\varphi(x))|. \end{aligned}$$

Thus, if  $X = \nabla(F \circ \rho \circ \varphi)$ , we have

$$\operatorname{div}_M X \geq 1 - \sup_M (F'(\rho_x) |\mathbf{H}(x)|) = 1 - A$$

and

$$|X| \leq F'(\rho_x) = I_m^{-1}(\rho_x) \leq \frac{1}{\inf_{[0, r_\varphi]} I_m(r)},$$

and we conclude that

$$\lambda^*(M) \geq (1 - A)^2 \inf_{[0, r_\varphi]} \frac{(I_m(r))^2}{4}.$$

The estimate

$$\lambda^*(M) \geq \frac{1 - A}{\int_0^{r_\varphi} I_m^{-1}(t) dt}$$

in (a) is an application of Barta's Theorem. We consider first the case where  $r_\varphi = +\infty$ , and assume that  $I_m^{-1} \in L^1([0, +\infty))$  for otherwise the estimate is trivial. Define

$$\tilde{F}(r) = \int_r^{+\infty} I_m^{-1}(t) dt,$$

and let  $u = \tilde{F} \circ \rho_N \circ \varphi$ . Then  $u$  is positive on  $M$  and, since

$$\tilde{F}'(r) = -I_m^{-1}(r) = -\frac{\int_0^r \sigma^{m-1} dt}{\sigma^{m-1}(r)} < 0 \text{ and } \tilde{F}''(r) = -1 - (m-1) \frac{\sigma'}{\sigma} \tilde{F}'(r),$$

a computation similar to that performed in the first part of the proof shows that

$$-\Delta_M u \geq 1 + |\mathbf{H}| \tilde{F}'(\rho_x) = 1 - \frac{|\mathbf{H}|}{I_m(r)} \geq 1 - A$$

and it follows from Barta's Theorem that

$$\lambda^*(M) \geq \inf_M \left( \frac{-\Delta u}{u} \right) \geq \frac{1 - A}{\int_0^{r_\varphi} I_m^{-1} dt},$$

as required.



The case where  $r_\varphi < +\infty$  is similar. Since to apply Barta's theorem we need  $u$  to be positive, we note that our assumptions imply that  $I_m$  is well defined and positive in  $[0, r_\varphi + \varepsilon]$  for every  $\varepsilon > 0$  sufficiently small. Next we let

$$\tilde{F}_\varepsilon(r) = \int_r^{r_\varphi + \varepsilon} I_m^{-1} dt,$$

and define  $u_\varepsilon$  accordingly. Arguing as above shows that

$$\lambda^*(M) \geq \inf_M \left( \frac{-\Delta u_\varepsilon}{u_\varepsilon} \right) \geq \frac{1 - A_\varepsilon}{\int_0^{r_\varphi + \varepsilon} I_m^{-1} dt},$$

where

$$A_\varepsilon = \sup_M |\tilde{F}'_\varepsilon(\rho_x) \mathbf{H}|.$$

The conclusion now follows letting  $\varepsilon \rightarrow 0$ .

Finally, if  $\varphi$  is proper,  $I_m^{-1}$  is integrable on  $[0, \infty)$  and  $A < 1$ , then the function  $-u$  is bounded, proper, and satisfies

$$\Delta(-u) \geq 1 - A > 0$$

on  $M$ . Therefore, in the terminology of [9], it is a weak maximum principle violating exhaustion function, and the discreteness of the spectrum of  $M$  follows from [9, Theorem 32].  $\square$

*Proof of Theorem 3.* We maintain the notation of the first part of the proof of Theorem 1, and let  $v = F \circ \rho \circ \varphi$ . Then  $v$  is bounded above by the assumption that  $I_m^{-1} \in L^1([0, r_\varphi])$  and, by (7), it satisfies

$$\Delta_M v \geq 1 - (F'(\rho_x) |\mathbf{H}(x)|) = 1 - \frac{|\mathbf{H}(x)|}{I_m(\rho_x)}.$$

Since  $M$  is assumed to be stochastically complete, by the weak maximum principle at infinity there exists a sequence  $\{x_n\}$  in  $M$  such that

$$\lim_n v(x_n) = \sup_M v \quad \text{and} \quad \liminf_n \Delta_M v(x_n) \leq 0.$$

Since  $F$  is increasing, this implies that  $\rho_{x_n} \rightarrow r_\varphi$ . In particular, if  $\varphi$  is unbounded, we conclude that

$$\liminf_{x \rightarrow +\infty} \left( 1 - \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \right) \leq 0, \quad \text{that is,} \quad \limsup_{x \rightarrow \infty} \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \geq 1.$$

In the case where  $\varphi$  is bounded, we can still conclude that

$$\inf_M \left( 1 - \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \right) \leq 0, \quad \text{that is,} \quad \sup_M \frac{|\mathbf{H}(x)|}{I_m(\rho_x)} \geq 1.$$

$\square$

## 5. IMMERSIONS INTO PRODUCTS

The aim of this section is to prove versions of Theorems 1 and 3 for immersions into a product manifold  $N \times L$ , where the factor  $N$  satisfies condition (1) in the Introduction. This clearly relaxes the curvature conditions imposed on the target manifold. As a counterpart, we need to strengthen the assumptions replacing conditions on  $\mathcal{J}_m$  and  $I_M$  with analogous conditions on  $\mathcal{J}_{m-l}$  and  $I_{m-l}$  respectively.

In some sense, we make up for the presence of the factor  $L$  by imposing more negative curvature conditions on the factor  $N$ .

**Theorem 8.** *Let  $\varphi$  be an isometric immersion of a complete Riemannian manifold  $M$  of dimension  $m$  into the product  $N \times \mathbb{R}^l$ , where  $L$  and  $N$  are complete Riemannian manifolds of dimension  $n$  and  $l$ , respectively. Assume that  $N$  has a pole  $p \in N$  and that its radial sectional curvature satisfies*

$$(1) \quad K_N(x) \leq -G(\rho_N(x))$$

where  $\rho_N(y) = \text{dist}_N(p, y)$  and  $G$  is a smooth, even function on  $\mathbb{R}$ . Assume that  $m \geq l + 1$  and that the solution  $\sigma$  of (2) satisfies  $\sigma' \geq 0$  on  $[0, r_{\pi_N \varphi}]$ , where  $\pi_N$  is the projection onto  $N$ . Suppose further that  $\mathcal{J}_{m-l}$  is nondecreasing in that interval, and that

$$A = \sup_M \frac{|\mathbf{H}|(x)}{I_{m-l}(\rho_N(\pi_N \varphi(x)))} \leq 1,$$

where  $\mathbf{H}$  is the mean curvature vector field of  $\varphi$ . Then

- (a) *The bottom of the spectrum of the Laplace-Beltrami operator of  $M$  satisfies the estimate*

$$\lambda^*(M) \geq \max \left\{ (1-A)^2 \inf_{[0, r_{\pi_N \varphi}]} \frac{I_{m-l}(r)^2}{4}, \frac{(1-A)}{\int_0^{r_{\pi_N \varphi}} I_{m-l}(t)^{-1} dt} \right\}$$

- (b) *If  $\pi_N \circ \varphi$  is proper,  $\int_0^{+\infty} I_{m-l}^{-1} dt < +\infty$  and  $A < 1$  then the spectrum of  $-\Delta_M$  is discrete.*

*Proof.* We continue to keep the notation of the proof of Theorem 1. Assuming that  $F$  is smooth and satisfies  $F' \geq 0$ , we consider the function  $F \circ \rho_N \circ \pi_N \circ \varphi$ , and argue as in Theorem 1 to obtain

$$\begin{aligned} \Delta(F \circ \rho \circ \pi \circ \varphi)(x) &\geq (F'' - F' \frac{\sigma'}{\sigma})(\rho_x) \sum_{i=1}^m \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 \\ &\quad + (F' \frac{\sigma'}{\sigma})(\rho_x) \sum_{i=1}^m |d\pi_N d\varphi(X_i)|^2 + F'(\rho_x) \langle \nabla^N \rho_N, d\pi_N \mathbf{H} \rangle. \end{aligned}$$

Choosing

$$F(r) = \int_0^r \frac{\int_0^t \sigma^{m-l-2}(s) ds}{\sigma^{m-l-2}(t)} dt,$$

and inserting the identities

$$F'(r) = \frac{\int_0^r \sigma^{m-l-1}(s) ds}{\sigma^{m-l-1}(r)} = I_{m-l-}^{-1}(r) > 0 \text{ and } F''(r) = 1 - (m-l-1) \frac{\sigma'}{\sigma} F'(r),$$

into the last inequality yields

$$\begin{aligned} \Delta_M(F \circ \rho \circ \pi \circ \varphi)(x) &\geq \left[ 1 - (m-l-1)F' \frac{\sigma'}{\sigma} \right] (\rho_x) \sum_{i=1}^m \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 \\ &\quad + (F' \frac{\sigma'}{\sigma}) (\rho_x) \sum_{i=1}^m |d\pi_N d\varphi(X_i)|^2 + F'(\rho_x) \langle \nabla^N \rho_N, d\pi_N \mathbf{H} \rangle. \end{aligned}$$

Since  $d\pi_N$  is the orthogonal projection onto  $T_z N$  which is of codimension  $l$  in  $T_{(z,u)}(N \times L)$ , and  $d\varphi(X_i)$  are  $m$ -orthonormal vectors, it is easy to verify that

$$\sum_{i=1}^m |d\pi_N d\varphi(X_i)|^2 \geq m-l = (m-l) |\nabla^N \rho_N|^2.$$

Using this, the fact that  $|\nabla^N \rho|^2 = 1 \geq \sum_i \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2$ , and the assumption that  $\mathcal{J}_{m-l}$  is nondecreasing in the form

$$(m-l) \frac{\sigma'}{\sigma} F'(r) \geq 1$$

we conclude that the right hand side of the above inequality is bounded below by

$$\begin{aligned} (m-l)F' \frac{\sigma'}{\sigma} (\rho_x) \left[ 1 - \sum_{i=1}^m \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 \right] + \sum_{i=1}^m \langle \nabla^N \rho_N, d\pi_N d\varphi X_i \rangle^2 - F'(\rho_x) |\mathbf{H}| \\ \geq 1 - F'(\rho_x) |\mathbf{H}| \end{aligned}$$

Thus, if  $X = \nabla(F \circ \rho_N \circ \pi_N \circ \varphi)$ , then

$$\operatorname{div}_M X \geq 1 - F'(\rho_x) |\mathbf{H}|,$$

and the estimate

$$(1-A)^2 \inf_{[0, r_{\pi_N \varphi}]} \frac{I_{m-l}(r)^2}{4}$$

follows arguing as in Theorem 1.

In a completely similar manner, the second estimate in a) follows applying Barta's theorem to the function  $u = \tilde{F} \circ \rho_N \circ \pi_N \circ \varphi$  with

$$\tilde{F}(r) = \int_r^{\operatorname{diam}(\pi_N \varphi(M))} I_{m-l}(t)^{-1} dt,$$

and conclusion b) in the statement is obtained noticing that if  $\pi_N \circ \varphi$  is proper, and  $I_{m-l}^{-1}$  is integrable, then  $-u$  is a weak maximum principle violating exhaustion function.  $\square$

In a similar fashion we have the following analogue of Theorem 3, which complements previous results by L. J. Alias, G.P. Bessa and M. Dacjzer, [1].

**Theorem 9.** *Let  $\varphi: M \rightarrow N$  be an isometric immersion of a stochastically complete,  $m$ -dimensional Riemannian manifold  $M$  into the product  $N \times L$ , where  $N$  and  $L$  are complete Riemannian manifolds of dimension  $n$  and  $l$  respectively, with  $m \geq l+1$ , and  $N$  satisfies the conditions listed in the statement of Theorem 8. Assume also that*

- (i)  $\mathcal{J}_{m-l}(r)$  is non-decreasing on  $[0, \pi_N(\varphi(M))]$ ,

(ii)  $I_{m-l}(r)^{-1} \in L^1(+\infty)$  if  $\pi_N \varphi$  is unbounded.

Then

$$\sup_M \frac{|\mathbf{H}(x)|}{I_{m-l}(\rho_x)} \geq 1$$

In particular, if  $r_{\pi_N \varphi} = +\infty$ , ( $\pi_N \varphi$  is unbounded in  $N$ ) then

$$\limsup_{x \rightarrow \infty} \frac{|\mathbf{H}(x)|}{I_{m-l}(\rho_x)} \geq 1.$$

If  $r_{\pi_N \varphi} = +\infty$  and  $I_{m-l}(r)^{-1} \rightarrow 0$  as  $r \rightarrow +\infty$  then

$$\sup_M |H| = +\infty.$$

We conclude this section noting that the above arguments can be used to give the following version for products of the already cited mean exit time comparison results obtained in [15].

**Theorem 10** (Stochastic estimates). *Let  $M$ ,  $N$  and  $L$  be complete Riemannian manifolds of dimensions  $m$ ,  $n$  and  $l$ , as in the statement of Theorem 9, with  $m \geq l + 1$ . Let  $\varphi : M \rightarrow N \times L$  be a minimal immersion, and assume that (i) and (ii) in the statement of Theorem 9 hold. Then  $M$  is not  $L^1$ -Liouville.*

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